

# Bäcklund transformations for the nonholonomic Veselova system

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## Abstract

We present auto and hetero Bäcklund transformations of the nonholonomic Veselova system using standard divisor arithmetic on the hyperelliptic curve of genus two. As a by-product one gets two natural integrable systems on the cotangent bundle to the unit two-dimensional sphere whose additional integrals of motion are polynomials in the momenta of fourth order.

## 1 Introduction

In nonholonomic mechanics a special attention is given to the systems whose equations of motion after suitable reduction yield conformally Hamiltonian vector field, which can be studied by the standard methods of Hamiltonian mechanics after an appropriate time reparameterization, see reviews [2, 6, 5, 9, 15] and the references therein. For instance, because the generic level set of the integrals of motion is independent of time, we can study symmetries of this manifold simultaneously for Hamiltonian and non-Hamiltonian vector fields associated with this level set.

The main our aim is to obtain Bäcklund transformations (BT) for the nonholonomic Veselova system [38]. Historically, the Bäcklund transformations originate in differential geometry in the 1880s and, in particular, they arose as certain transformations between surfaces. In the theory of integrable systems, according to classical definition by Darboux [12], a Bäcklund transformation (BT) between the two given PDEs

$$H(u, x, t) = 0 \quad \text{and} \quad \tilde{H}(\tilde{u}, \tilde{x}, \tau) = 0$$

is a pair of relations

$$F_{1,2}(u, x, t, \tilde{u}, \tilde{x}, \tau) = 0 \tag{1.1}$$

and some additional relations between  $(x, t)$  and  $(\tilde{x}, \tau)$ , which allow to get both equations  $H$  and  $\tilde{H}$ , see details in [1, 25]. The BT is called an auto-BT or a hetero-BT depending whether the two PDEs are the same or not. Auto BTs are used to generate solutions of the given partial differential equation, starting from known solutions, even trivial ones. The hetero BTs describe a correspondence between two different PDE's rather than a one-to-one mapping between their solutions.

In classical mechanics, auto BT of the Hamilton-Jacobi partial differential equation  $H = E$  is a canonical transformation of variables

$$\mathcal{B}: \quad (u, p_u) \rightarrow (\tilde{u}, \tilde{p}_u) \tag{1.2}$$

on the phase space  $\mathcal{M}$  preserving the form of this Hamilton-Jacobi equation [28, 39]. There are also other definitions associated with Lax pair [27] or with the corresponding algebraic curve [23], but we prefer to use definition related to the Hamilton-Jacobi equation itself. A comprehensive definition of hetero BT is also not yet available. Some examples the hetero BTs relating different Hamilton-Jacobi equations were considered in [26, 32, 33, 34].

For many integrable by quadratures dynamical systems in holonomic and nonholonomic mechanics one faces the Abel quadratures, which for the Veselova system with were obtained in original paper [38]. In these cases solutions of equations of motion are given by Jacobi inversion of Abel quadratures and the corresponding Abel map allows us to relate the generic level set of

integrals of motion with the Jacobian  $Jac(\mathcal{C})$  of some algebraic curve  $\mathcal{C}$ , which has a well-studied group structure. It also allows us to consider original variables  $(u, p_u)$  and their images  $(\tilde{u}, \tilde{p}_u)$  on  $\mathcal{M}$  as coordinates of two reduced divisors  $D$  and  $\tilde{D}$  on  $Jac(\mathcal{C})$  and to identify any auto BT (1.2) with a composition of suitable group operations

$$D \approx \tilde{D}, \quad D + D' = \tilde{D} \quad \text{and} \quad [\ell]D = \tilde{D}, \quad (1.3)$$

where  $\approx$ ,  $+$  and  $[\ell]$  denote equivalence, addition and scalar multiplication by an integer and  $D'$  is an auxiliary divisor depending on arbitrary parameters.

In [10] Cantor proposed a concrete algorithm for performing computations in Jacobian groups of hyperelliptic curves which consists of two stages: the composition stage, which generally outputs an unreduced divisor, and the reduction stage, which transforms the unreduced divisor into the unique reduced divisor. Now we have a lot of algorithms and their professional computer implementations for the divisor arithmetics on low, mid and high-genus curves, for generic divisor doubling, tripling etc, see references in [11, 20, 24].

In Hamiltonian mechanics, authors usually consider only full degree divisors  $D, \tilde{D}$  and weight one divisor  $D'$  in (1.3) that makes the reduction stage of Cantor's algorithm trivial. This partial group operation and the corresponding auto BTs have been studied from the different points of view in many publications, in particular see [14, 23, 27] and references therein. In fact, this very special construction of auto BTs was developed for simplest one-parametric discretization of original continuous systems.

Below we discuss auto BTs associated with addition of two generic divisors and with generic divisor doubling when reduction coincides with inversion. These auto BTs represent hidden symmetries of the level manifold which yields new canonical variables on original phase space and, therefore, we can use these new variables to construction of new integrable systems, i.e. to construction of hetero BTs. In our opinion it is natural to use different types of auto BTs for the different purposes.

## 1.1 Veselova system

Following [38], consider the motion of the rigid body with a fixed point under nonholonomic constraint

$$(\Omega, \gamma) = 0. \quad (1.4)$$

Here  $\Omega = (\Omega_1, \Omega_2, \Omega_3)$  is the vector of the angular velocity in the body frame,  $\gamma = (\gamma_1, \gamma_2, \gamma_3)$  is a unit vector which is fixed in a space frame,  $(x, y)$  and  $x \times y$  denote the scalar and vector products in  $\mathbb{R}^3$ , respectively. Isomorphism of this system with the system describing the motion of a nonhomogeneous ball without slipping and twisting on a plane was found in [7], see also [8].

A standard form of the equations of motion describing rotation of a rigid body around a fixed point is the following:

$$\frac{d}{d\tau}\gamma = \gamma \times \Omega, \quad \frac{d}{d\tau}M = M \times \Omega + \lambda\gamma. \quad (1.5)$$

The last term in the second equation in (1.5) is related to condition that projection of the angular velocity  $\Omega$  to a fixed vector  $\gamma$  must zero (1.4).

Here  $M = I\Omega$  is the vector of kinetic momentum of the body, expressed in the body frame. This frame is firmly attached to the body, its origin is in the body's fixed point, and its axes coincide with the principal inertia axes of the body. The inertia tensor of the body in this frame is diagonal

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad I_1, I_2, I_3 > 0.$$

Using the equations of motion (1.5) and constraint (1.4) we can calculate Lagrangian multiplier  $\lambda$

$$\lambda = \frac{(AM \times M, A\gamma)}{(A\gamma, \gamma)}, \quad A = I^{-1} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

Equations of motion (1.5) determine vector field  $Z$  on the manifold  $\mathcal{M} = \mathbb{R}^3 \times so^*(3)$  with coordinates  $z = (\gamma_1, \gamma_2, \gamma_3, M_1, M_2, M_3)$ :

$$\frac{d}{d\tau} z_i = Z_i, \quad i = 1, \dots, 6. \quad (1.6)$$

Vector field  $Z$  possesses four independent first integrals

$$H_1 = (M, \Omega), \quad H_2 = (M, M) - (\gamma, M)^2, \quad C_1 = (\gamma, \gamma) = 1, \quad C_2 = (\gamma, \Omega) = 0, \quad (1.7)$$

and an invariant measure

$$\mu = g \, d\gamma dM, \quad g = (\gamma, A\gamma)^{1/2} = \sqrt{a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2}. \quad (1.8)$$

Hence, it is integrable by quadratures according the Euler-Jacobi theorem [38].

In order to get these quadratures we introduce the following Poisson bivector on  $\mathcal{M}$

$$P = \begin{pmatrix} 0 & 0 & 0 & \frac{\gamma_1\gamma_2\gamma_3(a_3-a_2)}{g} & \gamma_3\left(g + \frac{(a_3-a_2)x_2^2}{g}\right) & -\gamma_2\left(g - \frac{(a_3-a_2)x_3^2}{g}\right) \\ 0 & 0 & 0 & -\gamma_3\left(g - \frac{(a_1-a_3)x_1^2}{g}\right) & \frac{\gamma_1\gamma_2\gamma_3(a_1-a_3)}{g} & \gamma_1\left(g + \frac{(a_1-a_3)x_3^2}{g}\right) \\ 0 & 0 & 0 & \gamma_2\left(g - \frac{(a_1-a_2)x_1^2}{g}\right) & -\gamma_1\left(g + \frac{(a_1-a_2)x_2^2}{g}\right) & \frac{\gamma_1\gamma_2\gamma_3(a_2-a_1)}{g} \\ * & * & * & 0 & \frac{b_3}{g} & -\frac{b_2}{g} \\ * & * & * & -\frac{b_3}{g} & 0 & \frac{b_1}{g} \\ * & * & * & \frac{b_2}{g} & -\frac{b_1}{g} & 0 \end{pmatrix}, \quad (1.9)$$

where vector  $b = (b_1, b_2, b_3)$  is equal to

$$b = (\gamma, \gamma)AM - (A\gamma \times \gamma) \times M = \begin{pmatrix} (\gamma, \gamma)a_1M_1 - \gamma_1((a_2-a_1)\gamma_2M_2 + (a_3-a_1)\gamma_3M_3) \\ (\gamma, \gamma)a_2M_2 - \gamma_2((a_1-a_2)\gamma_1M_1 + (a_3-a_2)\gamma_3M_3) \\ (\gamma, \gamma)a_3M_3 - \gamma_3((a_1-a_3)\gamma_1M_1 + (a_2-a_3)\gamma_2M_2) \end{pmatrix}.$$

This bivector  $P$  may be obtained using Chaplygin method of reducing multiplier [6, 15], Poisson reduction on Lie groups [16] or the Turiel deformations of the canonical Poisson structures [29].

It is easy to prove that vector field  $Z$  (1.6) is a conformally Hamiltonian vector field on the phase space  $\mathcal{M} = T^*\mathbb{S}^2$

$$Z = \frac{1}{2g} X, \quad X = PdH_1,$$

where Hamiltonian  $H_1$  is given by (1.7).

In the next Section we present auto and hetero Bäcklund transformations for these vector fields  $Z$  and  $X$  having a common level set of integrals of motion.

## 2 Abel equations and auto Bäcklund transformations

According [38] we can integrate original equations of motion (1.5) after change of time  $d\tau \rightarrow 2gdt$ , i.e. after transition from conformally Hamiltonian vector field  $Z$  to Hamiltonian vector field  $X$ . Indeed, let us introduce variables  $u_{1,2}$  and  $p_{u_{1,2}}$  using equations

$$\gamma_i = \sqrt{\frac{(u_1 - I_i)(u_2 - I_i)}{(I_j - I_i)(I_k - I_i)}}, \quad i \neq j \neq k \quad (2.1)$$

and

$$M_i = \frac{2}{gI_jI_k} \cdot \frac{\varepsilon_{ijk}\gamma_j\gamma_k(I_j - I_k)}{u_1 - u_2} \left( (I_i - u_1)p_{u_1} - (I_i - u_2)p_{u_2} \right), \quad (2.2)$$

where  $\varepsilon_{ijk}$  is a totally skew-symmetric tensor. In this variables Poisson bivector (1.9) has the following form

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

i.e. variables  $u_{1,2}$  and  $p_{u_{1,2}}$  are canonical variables

$$\{u_i, p_{u_j}\} = \delta_{i,j}, \quad \{u_1, u_2\} = \{p_{u_1}, p_{u_2}\} = 0. \quad (2.3)$$

We can identify  $u_{1,2}$  with the standard elliptic or spheroconical coordinates on the unit sphere

$$\frac{\gamma_1^2}{x - I_1} + \frac{\gamma_2^2}{x - I_2} + \frac{\gamma_3^2}{x - I_3} = \frac{(x - u_1)(x - u_2)}{(x - I_1)(x - I_2)(x - I_3)}, \quad I_k = \frac{1}{a_k} \quad (2.4)$$

The defining equation (2.4) should be interpreted as an identity with respect to  $x$ , and for each set of elliptic coordinates  $u_{1,2}$  it is possible to solve (2.4) for  $\gamma_i$  by calculating the residues at  $x = I_i$ . Notice also that (2.4) implies  $(\gamma, \gamma) = 1$  and  $I_1 < I_2 < I_3$ , so that the elliptic coordinate system is orthogonal, and the coordinates  $u_{1,2}$  take values only in the intervals

$$I_1 < u_1 < I_2 < u_2 < I_3.$$

In geometry using a simultaneous rescaling of the coordinates and the parameters,  $u_i \rightarrow au_i + b$  and  $I_i \rightarrow aI_i + b$ , it is always possible to take  $I_1 = 0$  and  $I_3 = 1$ . In mechanics  $I_k > 0$  are the some fixed momenta of inertia which can not be changed.

The corresponding momenta

$$p_{u_{1,2}} = \frac{1}{2g} \left( \frac{\gamma_1(\gamma_2 M_3 - \gamma_3 M_2)}{x - I_1} + \frac{\gamma_2(\gamma_3 M_1 - \gamma_1 M_3)}{x - I_2} + \frac{\gamma_3(\gamma_1 M_2 - \gamma_2 M_1)}{x - I_3} \right)_{x=u_{1,2}} \quad (2.5)$$

differ on standard expressions of momenta via angular momentum by the factor  $g = \sqrt{(\gamma, A\gamma)}$  in the denominator. It would do no harm us to identify phase space  $\mathcal{M}$  with the cotangent bundle  $T^*\mathbb{S}^2$  of the unit sphere [6, 15, 38].

In this canonical variables  $T^*\mathbb{S}^2$  original Hamiltonians (1.7) are equal to

$$H_1 = \frac{\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{\varphi(u_2)p_{u_2}^2}{u_2 - u_1} \quad \text{and} \quad H_2 = \frac{u_2\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{u_1\varphi(u_2)p_{u_2}^2}{u_2 - u_1}, \quad (2.6)$$

where

$$\varphi(u) = 4u(1 - a_1u)(1 - a_2u)(1 - a_3u). \quad (2.7)$$

To describe evolution of  $u_{1,2}$  with respect to  $H_{1,2}$  we use the canonical Poisson bracket (2.3) and expressions for  $H_{1,2}$  to obtain

$$\frac{du_1}{dt_1} = \{u_1, H_1\} = \frac{2\varphi(u_1)p_{u_1}}{u_1 - u_2}, \quad \frac{du_2}{dt_1} = \{u_2, H_1\} = \frac{2\varphi(u_2)p_{u_2}}{u_2 - u_1} \quad (2.8)$$

and

$$\frac{du_1}{dt_2} = \{u_1, H_2\} = \frac{2u_2\varphi(u_1)p_{u_1}}{u_1 - u_2}, \quad \frac{du_2}{dt_2} = \{u_2, H_2\} = \frac{2u_1\varphi(u_2)p_{u_2}}{u_2 - u_1}. \quad (2.9)$$

To solve the Hamilton-Jacobi equations  $H_{1,2} = h_{1,2}$  with respect to  $p_{u_{1,2}}$

$$p_{u_k}^2 = \varphi(u_k)^{-1}(h_1 u_k - h_2), \quad k = 1, 2.$$

and to substitute these expressions into (2.8-2.9) one gets standard Abel quadratures

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 2dt_2, \quad \frac{u_1 du_1}{\sqrt{f(u_1)}} + \frac{u_2 du_2}{\sqrt{f(u_2)}} = 2dt_1, \quad (2.10)$$

on the hyperelliptic curve  $\mathcal{C}$  of genus two defined by equation

$$\mathcal{C}: \quad y^2 = f(x), \quad f(x) = 4x(a_1x - 1)(a_2x - 1)(a_3x - 1)(h_1x - h_2). \quad (2.11)$$

Thus, we can identify the generic level set of the first integrals for the Veselova system with a symmetric product  $\mathcal{C} \times \mathcal{C}$  and later with Jacobi variety  $Jac(\mathcal{C})$  of hyperelliptic curve  $\mathcal{C}$  of genus two [38]. It is a starting point in the construction of the auto and hetero Bäcklund transformations of the Veselova system.

**Remark 1** Multidimensional generalizations of the Veselova problem of a nonholonomic rigid body motion and Abel quadratures for the corresponding Hamiltonian vector fields are discussed in [15]. We can directly apply these quadratures to construction of the Bäcklund transformations in the framework of the Abel theory [35].

## 2.1 Abel equations and group operations

Suppose that transformation of variables

$$\mathcal{B}: (u_1, u_2, p_{u_1}, p_{u_2}) \rightarrow (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2}) \quad (2.12)$$

preserves Hamilton equations (2.8-2.9) and the form of Hamiltonians (2.6). It means that new variables satisfy to algebraic equations

$$\begin{aligned} \frac{\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{\varphi(u_2)p_{u_2}^2}{u_2 - u_1} &= H_1 = \frac{\varphi(\tilde{u}_1)\tilde{p}_{u_1}^2}{\tilde{u}_1 - \tilde{u}_2} + \frac{\varphi(\tilde{u}_2)\tilde{p}_{u_2}^2}{\tilde{u}_2 - \tilde{u}_1} \\ \frac{u_2\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{u_1\varphi(u_2)p_{u_2}^2}{u_2 - u_1} &= H_2 = \frac{\tilde{u}_2\varphi(\tilde{u}_1)\tilde{p}_{u_1}^2}{\tilde{u}_1 - \tilde{u}_2} + \frac{\tilde{u}_1\varphi(\tilde{u}_2)\tilde{p}_{u_2}^2}{\tilde{u}_2 - \tilde{u}_1}, \end{aligned}$$

and differential equations

$$\frac{d\tilde{u}_1}{\sqrt{f(\tilde{u}_1)}} + \frac{d\tilde{u}_2}{\sqrt{f(\tilde{u}_2)}} = 2dt_2, \quad \frac{\tilde{u}_1 d\tilde{u}_1}{\sqrt{f(\tilde{u}_1)}} + \frac{\tilde{u}_2 d\tilde{u}_2}{\sqrt{f(\tilde{u}_2)}} = 2dt_1. \quad (2.13)$$

Subtracting (2.13) from (2.10) one gets two Abel differential equations

$$\begin{aligned} \omega_1(x_1, y_1) + \omega_1(x_2, y_2) + \omega_1(x_3, y_3) + \omega_1(x_4, y_4) &= 0, \\ \omega_2(x_1, y_1) + \omega_2(x_2, y_2) + \omega_2(x_3, y_3) + \omega_2(x_4, y_4) &= 0, \end{aligned} \quad (2.14)$$

where

$$x_{1,2} = u_{1,2}, \quad y_{1,2} = \varphi(u_{1,2})p_{u_{1,2}}, \quad x_{3,4} = \tilde{u}_{1,2}, \quad y_{3,4} = -\varphi(\tilde{u}_{1,2})\tilde{p}_{u_{1,2}} \quad (2.15)$$

and  $\omega_{1,2}$  form a base of holomorphic differentials on hyperelliptic curve  $\mathcal{C}$  of genus two

$$\omega_1(x, y) = \frac{dx}{y}, \quad \omega_2(x, y) = \frac{xdx}{y}.$$

Here we change sign of  $y_{3,4}$  by using standard hyperelliptic involution  $(x, y) \rightarrow (x, -y)$ . Historical perspective and modern geometric meaning of Abel differential equations are discussed in [18, 22].

**Remark 2** Abel equations are closely related with the group law in the Jacobian of the corresponding algebraic curve [10, 14, 20, 23, 35]. In fact, there are two main methods for deriving the group law in Jacobian of hyperelliptic curve: the algebraic method [24] based on Harley's formulation of Cantor's algorithm, and the geometric method using interpolation of points [11], which is based on Clebsch's geometric formulation of the Abel theorem. In both cases, we can either use an abstract modern language of algebraic geometry, which is somewhat difficult to understand experts in mechanics, physics, practical cryptography etc, either use some trivial geometric construction, which allow us to get the necessary explicit formulae.

For the low-genus hyperelliptic curves and the corresponding mechanical systems it is easy to explain all the necessary explicit formulae in the framework of the classical Abel theory, i.e. without the modern abstractions applicable to all the possible cases. Indeed, let us consider Abel differential equations (2.14) on the genus two hyperelliptic curve

$$\mathcal{C}: y^2 = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0. \quad (2.16)$$

According to Abel's idea, solutions  $(x_i, y_i)$  are abscissas and ordinates of the points of  $\mathcal{C}$  intersecting with the second plane curve defined by equation

$$y - P(x) = 0, \quad P(x) = b_3x^3 + b_2x^2 + b_1x + b_0. \quad (2.17)$$

In our case four points of intersection  $p_1 = (x_1, y_1), \dots, p_4 = (x_4, y_4)$  are solutions of Abel equations (2.14), whereas two remaining points  $p_{5,6}$  are arbitrary. Using this freedom we can suppose that (see Figure 1):

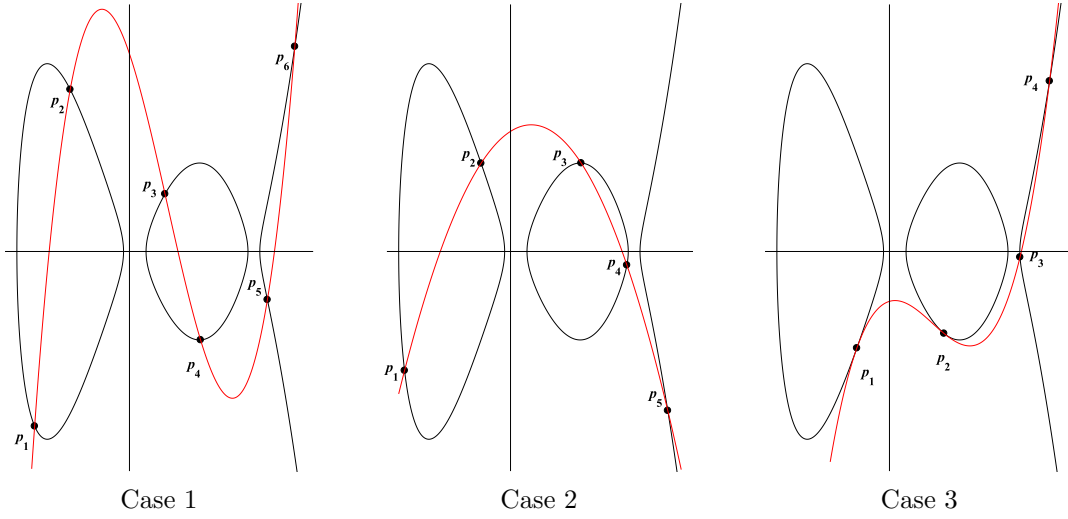


Figure 1: Standard picture from the paper on hyperelliptic curve cryptography [11].

1. points  $p_{5,6}$  are given at some fixed finite positions (three full degree divisors);
2. one point  $p_6$  is at infinity (two full degree divisors and divisor weight one);
3. points  $p_{5,6}$  coincides with  $p_{1,2}$  (generic divisor doubling).

**Remark 3** There are other reduced divisors on  $Jac(\mathcal{C})$  [10]. Here we consider only divisors associated with the given integrable system with two degrees of freedom (2.8-2.9). Examples of auto BTs associated with reduced divisors at Case 1 and at Case 2 may be found in [14, 27, 23]. Examples of auto BTs associated with a generic divisor doubling (Case 3) may be found in [36].

Substituting  $y = P(x)$  into the equation of the curve (2.16) we obtain so-called Abel polynomial

$$\psi(x) = P(x)^2 - f(x) = (b_3x^3 + b_2x^2 + b_1x + b_0)^2 - (a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0),$$

which has no multiple roots in the first and second cases and two double roots in the third case:

1.  $\psi(x) = b_3^2(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6),$
2.  $\psi(x) = -a_5(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5), \quad b_3 = 0.$
3.  $\psi(x) = b_3^2(x - x_1)^2(x - x_2)^2(x - x_3)(x - x_4).$

Using Abel polynomial  $\psi$  we can easily determine abscissas  $x_{3,4}$  as functions on coordinates of other points of intersection:

1.  $(x - x_3)(x - x_4) = \frac{\psi(x)}{b_3^2(x - x_1)(x - x_2)(x - x_5)(x - x_6)};$
2.  $(x - x_3)(x - x_4) = \frac{\psi(x)}{a_5(x - x_1)(x - x_2)(x - x_5)},$
3.  $(x - x_3)(x - x_4) = \frac{\psi(x)}{b_3^2(x - x_1)^2(x - x_2)^2}.$

Equating coefficients is an efficient way to compute the exact division required above gives:

$$1. \quad x_3 + x_4 = \frac{a_5 - 2b_2b_3}{b_3^2} - x_1 - x_2 - x_5 - x_6, \quad x_3x_4 = \frac{2b_1b_3 + b_2^2 - a_4}{b_3^2} - (x_1 + x_2 + x_5 + x_6)(x_3 + x_4) - x_1(x_2 + x_5 + x_6) - x_2(x_5 + x_6) - x_5x_6 \quad (2.18)$$

and similar for the second and third cases

$$\begin{aligned}
2. \quad x_3 + x_4 &= -x_1 - x_2 - x_5 + \frac{b_2^2 - a_4}{a_5}, \\
x_3 x_4 &= \frac{a_3 - 2b_1 b_2}{a_5} - (x_1 + x_2 + x_5)(x_3 + x_4) - x_1 x_2 - x_1 x_5 - x_2 x_5; \\
3. \quad x_3 + x_4 &= -2x_1 - 2x_2 + \frac{a_5 - 2b_2 b_3}{b_3^2}, \\
x_3 x_4 &= \frac{2b_1 b_3 + b_2^2 - a_4}{b_3^2} - 2(x_1 + x_2)(x_3 + x_4) - x_1^2 - 4x_1 x_2 - x_2^2.
\end{aligned} \tag{2.19}$$

Four coefficients  $b_3, b_2, b_1$  and  $b_0$  of the polynomial  $P(x)$  (2.17) are calculated by solving four algebraic equations:

$$\begin{aligned}
1. \quad y_{1,2} &= P(x_{1,2}), \quad y_{5,6} = P(x_{5,6}); \\
2. \quad y_{1,2} &= P(x_{1,2}), \quad y_5 = P(x_5), \quad b_3 = 0; \\
3. \quad y_{1,2} &= P(x_{1,2}), \quad \left. \frac{dP(x)}{dx} \right|_{x=x_{1,2}} = \left. \frac{d\sqrt{f(x)}}{dx} \right|_{x=x_{1,2}} \equiv \frac{1}{2y_{1,2}} f'(x_{1,2}),
\end{aligned} \tag{2.20}$$

where  $f'(x)$  is derivative of  $f(x)$  by  $x$ . Substituting coefficients  $b_k$  into (2.18-2.19) one gets abscissas  $x_{3,4}$  as functions on  $x_{1,2}, y_{1,2}$  and  $x_{5,6}, y_{5,6}$ . The corresponding ordinates  $y_{3,4}$  are equal to

$$y_{3,4} = P(x_{3,4}), \tag{2.21}$$

where polynomial  $P(x)$  is given by

$$\begin{aligned}
1. \quad P(x) &= \frac{(x - x_6)(x - x_5)(x - x_2)y_1}{(x_1 - x_5)(x_1 - x_6)(x_1 - x_2)} + \frac{(x - x_6)(x - x_5)(x - x_1)y_2}{(x_2 - x_5)(x_2 - x_6)(x_1 - x_2)} \\
&+ \frac{(x - x_6)(x - x_2)(x - x_1)y_5}{(x_5 - x_1)(x_5 - x_2)(x_6 - x_5)} + \frac{(x - x_5)(x - x_2)(x - x_1)y_6}{(x_6 - x_1)(x_6 - x_2)(x_6 - x_5)}, \\
2. \quad P(x) &= \frac{y_1(x - x_2)(x - \lambda)}{(x_1 - x_2)(x_1 - x_5)} + \frac{y_2(x - x_1)(x - x_5)}{(x_2 - x_1)(x_2 - x_5)} + \frac{y_5(x - x_1)(x - x_2)}{(x_5 - x_1)(x_5 - x_2)}, \\
3. \quad P(x) &= \frac{(x - x_2)^2(2x - 3x_1 + x_2)y_1}{(x_2 - x_1)^3} + \frac{(x - x_1)^2(2x + x_1 - 3x_2)y_2}{(x_1 - x_2)^3} \\
&+ \frac{(x - x_2)^2(x - x_1)f'(x_1)}{2(x_1 - x_2)^2 y_1} + \frac{(x - x_1)^2(x - x_2)f'(x_2)}{2(x_1 - x_2)^2 y_2}.
\end{aligned} \tag{2.22}$$

These simple equations (2.18-2.19) and (2.21) determine group operations relating points of intersection  $p_{1,2} = (x_{1,2}, y_{1,2})$  with  $p_{3,4} = (x_{3,4}, y_{3,4})$ .

**Remark 4** In generic cases we can take any professional implementation of Cantor's algorithm [10, 20] or implementations of various improvements and extensions to Cantor's algorithm for mid- or high-genus curves with general divisor doubling, tripling etc [11, 20].

## 2.2 Auto Bäcklund transformations

We identify variables on the phase space  $T^*\mathbb{S}^2$  with abscissas and ordinates of the points of intersection in (2.15)

$$x_{1,2} = u_{1,2}, \quad y_{1,2} = \varphi(u_{1,2})p_{u_{1,2}}, \quad x_{3,4} = \tilde{u}_{1,2}, \quad y_{3,4} = -\varphi(\tilde{u}_{1,2})\tilde{p}_{u_{1,2}}.$$

Following [15, 23, 27] we also denote coordinates of the remaining two points of intersection as follows

$$p_5 = (x_5, y_5) \equiv (\lambda_1, \mu_1), \quad p_6 = (x_6, y_6) \equiv (\lambda_2, \mu_2).$$

Abscissas  $\lambda_{1,2}$  are arbitrary numerical parameters, whereas  $\mu_{1,2} = \sqrt{f(\lambda_{1,2})}$  are non trivial combinations of the integrals of motion, i.e. nontrivial functions on the phase space.

In order to get explicit expressions for  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$  in terms of original elliptic coordinates  $u_{1,2}$  and momenta  $p_{u_{1,2}}$  we have to:

- find coefficients  $a_5, \dots, a_0$  from the separation relations (2.11, 2.16);
- calculate coefficients  $b_3, \dots, b_0$  from (2.20);
- substitute these coefficients into (2.18-2.19) and solve the resulting equations with respect to  $x_{3,4} = \tilde{u}_{1,2}$ ;
- calculate momenta  $\tilde{p}_{u_{1,2}}$  substituting  $\tilde{u}_{1,2}$  into (2.21)

$$y_{3,4} = P(x_{3,4}) \implies \tilde{p}_{u_{1,2}} = -\frac{P(\tilde{u}_{1,2})}{\varphi(\tilde{u}_{1,2})}.$$

In order to get canonical transformation  $\mathcal{B}$  (2.12) in term of original variables  $\gamma$  and  $M$  we have to substitute  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$  into (2.1) and (2.2) instead of  $u_{1,2}$  and  $p_{u_{1,2}}$  and use (2.5-2.5). We can easily obtain the desired explicit formulae for  $\tilde{\gamma}$  and  $\tilde{M}$  using any modern computer algebra system and, therefore, we do not show these bulky expressions here.

**Remark 5** We can avoid computer calculations and obtain the same expressions by hand using known Lax representations for finite dimensional systems [23, 27]. However, it is easy to prove that in fact Darboux transformations of the Lax pairs yield only implicit expressions for the images of original variables even for the systems with two and three degrees of freedom.

Using explicit formulae for the transformations  $\mathcal{B}$  we can prove the following statement.

**Proposition 1** *Equations (2.18-2.19) and (2.21) determine canonical transformations  $\mathcal{B}$  (2.12) on  $T^*\mathbb{S}^2$  of valency one and two for which original Poisson bracket (2.3) has the following form in new variables*

$$1, 2. \quad \{\tilde{u}_i, \tilde{p}_{u_j}\} = \delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0$$

$$3. \quad \{\tilde{u}_i, \tilde{p}_{u_j}\} = 2\delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0,$$

*respectively. These canonical transformations preserve the form of integrals of motion (1.7, 2.6), i.e. they are auto Bäcklund transformations for the Veselova system.*

The proof is a straightforward calculation.

Remind, that canonical transformation  $(u, p_u) \rightarrow (\tilde{u}, \tilde{p}_u)$  of valency  $c$  determines the Jacobi matrix

$$V = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial p_u} \\ \frac{\partial \tilde{p}_u}{\partial u} & \frac{\partial \tilde{p}_u}{\partial p_u} \end{pmatrix},$$

which is a generalized symplectic matrix of valence  $c$

$$V^\top \Omega V = c\Omega, \quad \Omega = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix},$$

see [17].

In the first and second cases integrable map  $\mathcal{B}$  determines exact-time discretization of continuous Hamiltonian flow depending on two parameters  $\lambda_{1,2}$  or one parameter  $\lambda_1$ , respectively. Because trajectories of the Hamiltonian vector field  $X$  and original conformally Hamiltonian vector field  $Z = \rho X$  coincide to each other, the same map  $\mathcal{B}$  can be considered also as discretization



of the original nonholonomic Veselova system. Another discretization of the Veselova system was obtained in the framework of Lagrange formalism in [19].

In the third case  $\mathcal{B}$  is a hidden symmetry of the generic level set of the integrals for the Veselova system which is some counterpart of the usual Noether symmetries. Of course, we can not apply such auto BT to discretization because when one iterates such BT one will obtain again original elliptic coordinates up to the factor 2. Other hidden symmetries can be obtained fixing values of  $\lambda_{1,2}$  at the first and second cases.

For  $\lambda = 0$  and  $\mu = \sqrt{f(\lambda)} = 0$  we have canonical transformation  $\mathcal{B}$  (2.12) on  $T^*\mathbb{S}^2$  which in term of the original coordinates have the following form

$$\tilde{\gamma}_i = \sqrt{\frac{a_j M_j^2 + a_k M_k^2}{a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2} - \frac{a_i \gamma_i^2}{g^2}}, \quad i \neq j \neq k$$

$$\tilde{M}_i = \frac{1}{g \tilde{g} \tilde{\gamma}_j \tilde{\gamma}_k} (\gamma_i (a_j \gamma_k \tilde{\gamma}_j^2 M_j + a_k \gamma_j \tilde{\gamma}_k^2 M_k) - (a_j \tilde{\gamma}_j^2 + a_k \tilde{\gamma}_k^2) \gamma_j \gamma_k M_i).$$

Here  $g = \sqrt{a_1 \gamma_1^2 + a_2 \gamma_2^2 + a_3 \gamma_3^2}$  and  $\tilde{g} = \sqrt{a_1 \tilde{\gamma}_1^2 + a_2 \tilde{\gamma}_2^2 + a_3 \tilde{\gamma}_3^2}$  are the last Jacobi multipliers related to each other via integrals of motion:

$$g \cdot \tilde{g} = \sqrt{\frac{a_1 a_2 a_3 H_2}{H_1}}.$$

It is easy to prove that this transformation  $\mathcal{B} : (\gamma, M) \rightarrow (\tilde{\gamma}, \tilde{M})$  preserve the form of the Poisson bivector  $P$  (1.9) and the form of integrals of motion, i.e.

$$C_1 = (\gamma, \gamma) = (\tilde{\gamma}, \tilde{\gamma}) = 1, \quad C_2 = (\gamma, AM) = (\tilde{\gamma}, A\tilde{M}) = 0,$$

and

$$H_1 = (M, AM) = (\tilde{M}, A\tilde{M}), \quad H_2 = (M, M) - (\gamma, M)^2 = (\tilde{M}, \tilde{M}) - (\tilde{\gamma}, \tilde{M})^2.$$

Below we use this hidden symmetry of the level set manifold to construction of new integrable systems on  $T^*\mathbb{S}^2$ .

### 3 Hetero Bäcklund transformations and new integrable systems on the sphere

The counterpart of the hetero Bäcklund transformations for finite dimensional integrable systems has to be a canonical transformation, which has to relate two different systems of the Hamilton-Jacobi equations

$$H_i \left( u, \frac{\partial S}{\partial u} \right) = h_i \quad \text{and} \quad \tilde{H}_i \left( \tilde{u}, \frac{\partial \tilde{S}}{\partial \tilde{u}} \right) = \tilde{h}_i \quad (3.1)$$

and has to satisfy some additional conditions which allow to get a non-trivial, usable and efficient theory. In [32, 33, 34] we postulated that  $H_i$  are simultaneously separable in  $u$  and  $\tilde{u}$  variables. This allows us to apply the standard Jacobi method to construction of  $\tilde{H}_i$ :

1. take Hamilton-Jacobi equation  $H = E$  separable in variables  $u, p_u$ ;
2. make auto Bäcklund transformation of variables  $(u, p_u) \rightarrow (\tilde{u}, \tilde{p}_u)$ , which conserves not only the Hamiltonian character of the equations of motion, but also the form of Hamilton-Jacobi equation;
3. substitute new canonical variables  $\tilde{u}, \tilde{p}_u$  into the suitable separated relations and obtain new integrable systems.

Step two is a simple technical exercise in the framework of the Abel theory due to its various implementations in modern cryptography [10].

### 3.1 First bi-Hamiltonian system

Integrals of motion (2.6) satisfy separated relations

$$\varphi(u_k)p_{u_k}^2 = H_1 u_k - H_2, \quad k = 1, 2, \quad (3.2)$$

and similar in new variables

$$\varphi(\tilde{u}_k)\tilde{p}_{u_k}^2 = H_1 \tilde{u}_k - H_2, \quad k = 1, 2, \quad (3.3)$$

i.e.  $H_{1,2}$  are simultaneously separable in  $u$  and  $\tilde{u}$ -variables. If we substitute new canonical variables  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$  on  $T^*\mathbb{S}^2$  into the following separation relations

$$2v_1 \equiv \varphi(\tilde{u}_1) \cdot \tilde{p}_{u_1}^2 = \tilde{H}_1 + \tilde{H}_2, \quad 2v_2 \equiv \varphi(\tilde{u}_2) \cdot \tilde{p}_{u_2}^2 = \tilde{H}_1 - \tilde{H}_2 \quad (3.4)$$

and solve the resulting equations with respect to  $\tilde{H}_{1,2}$

$$\tilde{H}_1 = v_1 + v_2 = \frac{(\tilde{u}_1 + \tilde{u}_2)H_1}{2} - H_2, \quad \tilde{H}_2 = v_1 - v_2 = \frac{(\tilde{u}_1 - \tilde{u}_2)H_1}{2}.$$

one gets some additive deformation  $\tilde{H}_1 = -H_2 + \Delta H$  of the second integral of motion  $H_2$  in (2.6).

**Remark 6** These integrals  $\tilde{H}_{1,2}$  can be obtained using definitions (2.19) of the symmetric functions on  $x_{3,4} = \tilde{u}_{1,2}$  without calculations of much more complicated expressions for  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$ .

These new Hamiltonians  $\tilde{H}_{1,2}$  are in involution with respect to compatible Poisson brackets

$$\{\tilde{u}_i, \tilde{p}_{u_j}\} = \delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0,$$

and

$$\{\tilde{u}_i, \tilde{p}_{u_j}\}' = \lambda_i^{-1} \delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\}' = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\}' = 0.$$

Using the corresponding bivectors  $P$  and  $P'$  it is easy to prove that vector field

$$X = Pd(v_1 + v_2) = P'd\left(\frac{v_1^2 + v_2^2}{2}\right)$$

is bi-Hamiltonian vector field. This trivial in  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$  variables Hamiltonian  $\tilde{H} = \tilde{H}_1$  has more complicated form in original variables:

$$\tilde{H} = v_1 + v_2 = \frac{1}{g^2(u_1 - u_2)^2} \left( \eta_1 \varphi(u_1) p_{u_1}^2 + \frac{\varphi(u_1) \varphi(u_2) p_{u_1} p_{u_2}}{2} + \eta_2 \varphi(u_2) p_{u_2}^2 \right), \quad (3.5)$$

where  $\varphi(u)$  is given by (2.7),

$$\eta_1 = u_2 + u_1 u_2 (u_2 \alpha_2 - \alpha_1 - (2u_2 - u_1) u_2 \alpha_3),$$

$$\eta_2 = u_1 + u_1 u_2 (u_1 \alpha_2 - \alpha_1 - (2u_1 - u_2) u_1 \alpha_3)$$

and

$$\alpha_1 = a_1 + a_2 + a_3, \quad \alpha_2 = a_1 a_2 + a_1 a_3 + a_2 a_3, \quad \alpha_3 = a_1 a_2 a_3.$$

Second integral of motion  $\tilde{H}_2 = v_1 - v_2$  is an algebraic function and, therefore, we present another function on  $\lambda_{1,2}$  which is the polynomial of fourth order in momenta

$$\tilde{K} = 4v_1 v_2 = \left( \frac{u_1 p_{u_1} - u_2 p_{u_2}}{u_1 - u_2} \right)^2 \cdot \frac{\varphi(u_1) \varphi(u_2)}{u_1 u_2 (u_1 - u_2)} \left( \frac{\varphi(u_1) p_{u_1}^2}{u_1} - \frac{\varphi(u_2) p_{u_2}^2}{u_2} \right). \quad (3.6)$$

Integral of motion  $\tilde{K}$  is factored on two polynomials of second order in momenta which do not commute with  $\tilde{H}$ . In three dimensional case similar examples of quartic integrals of motion are discussed in [31].

In redundant variables  $\gamma$  and  $M$  these Hamiltonians have the following form:

$$\tilde{H} = g^{-2}(M, \Omega) - (M, M) + 2(\gamma, M)^2, \quad \tilde{K} = 4a_1 a_2 a_3 (\gamma, M)^2 \cdot ((M, M) - (\gamma, M)^2). \quad (3.7)$$

**Proposition 2** *If  $(\gamma, \gamma) = 1$  and  $(\gamma, \Omega) = 0$ , Hamiltonians (3.7) are in involution*

$$\{\tilde{H}, \tilde{K}\} = 0$$

*with respect to the Poisson bracket defined by the Poisson bivector  $P$  (1.9).*

The proof is a straightforward calculation.

Using more complicated separation relations instead of (3.4) we can get more complicated Hamiltonians on  $T^*\mathbb{S}^2$  which have a natural form  $\tilde{H} = T + V$  and an additional quartic integral of motion [32, 33, 34, 35].

### 3.2 Second bi-Hamiltonian system

Let us rewrite separated relations (3.3) for original integrals of motion (2.6) in the following form

$$\frac{\varphi(\tilde{u}_k)\tilde{p}_{u_k}^2}{\tilde{u}_k} = H_1 - \frac{H_2}{u_k}, \quad k = 1, 2.$$

If we substitute new canonical variables  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$  on  $T^*\mathbb{S}^2$  into the following separation relations

$$2\nu_1 \equiv \frac{\varphi(\tilde{u}_1)\tilde{p}_{u_1}^2}{u_1} = \hat{H}_1 + \hat{H}_2, \quad 2\nu_2 \equiv \frac{\varphi(\tilde{u}_2)\tilde{p}_{u_2}^2}{u_2} = \hat{H}_1 - \hat{H}_2 \quad (3.8)$$

and solve the resulting equations with respect to  $\hat{H}_{1,2}$

$$\hat{H}_1 = \nu_1 + \nu_2 = H_1 - \frac{(\tilde{u}_1 + \tilde{u}_2)H_2}{2\tilde{u}_1\tilde{u}_2}, \quad \hat{H}_2 = \nu_1 - \nu_2 = \frac{(\tilde{u}_1 - \tilde{u}_2)H_2}{2\tilde{u}_1\tilde{u}_2}$$

one gets some additive deformation  $\hat{H}_1 = H_1 - \Delta H_1$  of the first integral of motion  $H_1$  in (2.6).

These new Hamiltonians  $\hat{H}_{1,2}$  are in involution with respect to compatible Poisson brackets

$$\{\tilde{u}_i, \tilde{p}_{u_j}\} = \delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0,$$

and

$$\{\tilde{u}_i, \tilde{p}_{u_j}\}'' = \nu_i^{-1}\delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\}'' = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\}'' = 0.$$

Using the corresponding bivectors  $P$  and  $P''$  it is easy to prove that vector field

$$X = Pd(\nu_1 + \nu_2) = P''d\left(\frac{\nu_1^2 + \nu_2^2}{2}\right)$$

is bi-Hamiltonian vector field. This trivial in  $\tilde{u}_{1,2}$  and  $\tilde{p}_{u_{1,2}}$  variables Hamiltonian  $\hat{H} = \hat{H}_1$  has more complicated form in original variables:

$$\hat{H} = \nu_1 + \nu_2 = \frac{1}{(u_1 - u_2)^2} \left( \zeta_1 \varphi(u_1) p_{u_1}^2 + \frac{\varphi(u_1)\varphi(u_2)p_{u_1}p_{u_2}}{2} + \zeta_2 \varphi(u_2) p_{u_2}^2 \right), \quad (3.9)$$

where  $\varphi(u)$  is given by (2.7),

$$\zeta_1 = 2u_1 - u_2 + u_1 u_2 (\alpha_2 u_2 - \alpha_1) - \alpha_3 u_1^2 u_2^2,$$

$$\zeta_2 = 2u_2 - u_1 + u_1 u_2 (\alpha_2 u_1 - \alpha_1) - \alpha_3 u_1^2 u_2^2.$$

Second integral of motion  $\hat{H}_2 = \nu_1 - \nu_2$  is an algebraic function and, therefore, we present another function on  $\nu_{1,2}$  which is the polynomial of fourth order in momenta

$$\hat{K} = 4\nu_1\nu_2 = \left( \frac{u_1 p_{u_1} - u_2 p_{u_2}}{u_1 - u_2} \right)^2 \cdot \frac{\varphi(u_1)\varphi(u_2)}{u_1 u_2 (u_1 - u_2)} (\varphi(u_1) p_{u_1}^2 - \varphi(u_2) p_{u_2}^2). \quad (3.10)$$

Integral of motion  $\hat{K}$  is factored on two polynomials of second order in momenta and the first factor coincides with the first factor in  $\tilde{K}$  (3.6).

In redundant variables  $\gamma$  and  $M$  these Hamiltonians have the following form:

$$\hat{H} = (M, \Omega) - g^2(M, M) = (M, \hat{\Omega}), \quad \hat{K} = 4g^2(\gamma, M)^2(M, \Omega), \quad (3.11)$$

where  $\hat{\Omega} = (A - (\gamma, A\gamma))M$  is the vector of "angular velocity" depending on  $\gamma$  variables.

**Proposition 3** *If  $(\gamma, \gamma) = 1$  and  $(\gamma, \Omega) = 0$ , Hamiltonians (3.11) are in involution*

$$\{\hat{H}, \hat{K}\} = 0$$

*with respect to the Poisson bracket defined by the Poisson bivector  $P$  (1.9).*

The proof is a straightforward calculation.

The corresponding conformally Hamiltonian vector field

$$\hat{Z} = \frac{1}{2g} Pd\hat{H} = \frac{1}{2g} Pd(H + \Delta H) = Z + \Delta Z$$

is additive deformation of the original vector field  $Z$  (1.5) for the Veselova system:

$$\begin{aligned} \frac{d}{d\tau}\gamma &= \gamma \times \hat{\Omega} - (\gamma, M) \cdot \gamma \times A\gamma, \\ \frac{d}{d\tau}M &= M \times \hat{\Omega} + \lambda\gamma + \frac{1}{2} \cdot \gamma \times \frac{\partial \hat{H}}{\partial \gamma} + (M - (\gamma, M)\gamma) \times AM. \end{aligned} \quad (3.12)$$

Such nonstandard equations of motion may appear in the study of a wide range of fields such as control theory, seismology, biological systems, in the study of a self gravitating stellar gas cloud, optoelectronics, fluid mechanics etc.

**Remark 7** By adding to the Hamiltonian  $H_1$  (1.7) one of the integrable potentials  $V(\gamma)$  from [13, 15] we change equation of motion

$$\frac{d}{d\tau}\gamma = \gamma \times \Omega, \quad \frac{d}{d\tau}M = M \times \Omega + \frac{1}{2} \cdot \gamma \times \frac{\partial V}{\partial \gamma},$$

and obtain new canonical variables on  $T^*\mathbb{S}^2$  after suitable Bäcklund transformation. For instance, if  $V = b(I_1x_1^2 + I_2x_2^2 + I_3x_3^2)$  [38], we have to change original separation relations (3.2) on

$$\varphi(u_k)p_{u_k}^2 = bu_k^2 + u_kH'_1 - H'_2, \quad b \in \mathbb{R}.$$

After Bäcklund transformation for this system one gets new canonical variables on  $T^*\mathbb{S}^2$  depending on parameter  $b$ . Substituting these variables into the separation relations (3.8) we obtain Hamiltonian

$$\hat{H}' = \hat{H} + \frac{4\sqrt{b}\sqrt{a_1a_2a_3}}{u_1 - u_2} (u_2^2\varphi(u_1)p_{u_1} - u_1^2\varphi(u_2)p_{u_2}) + b(\alpha_3u_1^2 + u_2^2 - \alpha_1u_1u_2 + 2u_1 + 2u_2), \quad (3.13)$$

with potential depending on velocities. In this case second integrals of motion is the polynomial of fourth order in momenta which does not factorized on two polynomials of second order, see other similar examples in [33, 34].

### 3.3 Bi-Hamiltonian systems on the two-dimensional sphere

A special class of natural Hamiltonian systems are geodesic flows, i.e., natural Hamiltonian systems with zero potential. According to the Maupertuis principle, an integrable natural Hamiltonian system immediately gives a family of integrable geodesic flows. If the integral of the system is polynomial in momenta, the integrals of the geodesic flows are also polynomial of the same degree. There are a few examples of integrable geodesic flows on the sphere  $\mathbb{S}^2$  whose integrals are polynomials in momenta of degree four, see [3, 4, 21, 37, 40] and references within.

Using the following anzats for integrals of motion on  $T^*\mathbb{S}^2$

$$\begin{aligned} H &= \frac{f_1(u_1, u_2)p_{u_1}^2 + f_2(u_1, u_2)p_{u_1}p_{u_2} + f_3(u_1, u_2)p_{u_2}^2}{(u_1 - u_2)^2}, \\ K &= \left( \frac{u_1p_{u_1} - u_2p_{u_2}}{u_1 - u_2} \right)^2 \left( h_1(u_1, u_2)p_{u_1}^2 + h_2(u_1, u_2)p_{u_2}^2 \right), \end{aligned}$$

where  $f_i$  and  $h_k$  are arbitrary functions on elliptic coordinates  $u_{1,2}$  on the sphere, we can prove that equation  $\{H, K\} = 0$  has two complete solutions (3.5-3.6) and (3.9-3.10). There is also one partial solution at  $a_1 = a_2$  associated with the Kovalevskaya systems in the dynamics of a rigid body.

Let us consider the two dimensional unit sphere  $\mathbb{S}^2$  as an imbedded manifold in  $\mathbb{R}^3$

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3; \quad (x, x) = 1\}.$$

Hence its cotangent bundle  $T^*\mathbb{S}^2$  can be realized as a subvariety of  $T^*\mathbb{R}^3$

$$T^*\mathbb{S}^2 = \{(x, p) \in T^*\mathbb{R}^3; \quad (x, x) = 1, \quad (p, x) = 0\}.$$

Here  $x = (x_1, x_2, x_3)$  and  $p = (p_1, p_2, p_3)$  are canonical coordinates in  $T^*\mathbb{R}^3$ .

We also will use coordinates  $x = (x_1, x_2, x_3)$  and  $J = (J_1, J_2, J_3)$  on the Euclidean algebra  $e(3)^*$  with the Lie-Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0, \quad (3.14)$$

where  $\varepsilon_{ijk}$  is the totally skew-symmetric tensor. Fixing values

$$(x, x) = 1, \quad (x, J) = 0$$

of the Casimir functions one gets symplectic leaf of  $e^*(3)$  which is a four-dimensional symplectic manifold equivalent to  $T^*\mathbb{S}^2$  [3].

**Remark 8** Variables  $(x, J)$  are related with the original Veselova variables by the following transformation [6]:

$$x = g^{-1} I^{-1/2} \gamma, \quad J = g I^{1/2} \Omega,$$

which reduce Poisson bivector (1.9) to the standard Poisson bivector associated to (3.14)

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 0 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & 0 & x_2 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 & J_3 & -J_2 \\ -x_3 & 0 & x_1 & -J_3 & 0 & J_1 \\ x_2 & -x_1 & 0 & J_2 & -J_1 & 0 \end{pmatrix},$$

up to the factor  $\sqrt{a_1 a_2 a_3}$ .

Let us rewrite Hamiltonians (3.5-3.6) and (3.9-3.10) in term of redundand variables  $(x, p)$  and  $(x, J)$  on  $T^*\mathbb{R}^3$ .

**Proposition 4** *If  $A$  is an arbitrary symmetric matrix defining two functions on  $T^*\mathbb{R}^3$*

$$\begin{aligned} \hat{H} &= (x, x)(J, AJ) - (x, Ax)(J, J) = \begin{vmatrix} (x, x) & (x, Ax) \\ (J, J) & (J, AJ) \end{vmatrix} \\ &= (x \times p, A(x \times p)) - (x, Ax)p^2 \end{aligned} \quad (3.15)$$

and

$$\hat{K} = (x, AJ)^2 (J, J) = (Ax, x \times p)p^2 = [x, p, Ax]^2 p^2,$$

where  $[x, p, Ax]^2$  is a Gram determinant

$$[x, p, Ax]^2 = \begin{vmatrix} (x, x) & (x, p) & (x, Ax) \\ (x, p) & (p, p) & (p, Ax) \\ (x, Ax) & (p, Ax) & (Ax, Ax) \end{vmatrix},$$

then the Poisson brackets (3.14) between  $\hat{H}$  and  $\hat{K}$  is equal to

$$\{\hat{H}, \hat{K}\} = (x, J)(x, AJ)(J, J) \cdot \{(J, AJ), (x, Ax)\}.$$

Hence,  $\hat{H}$  and  $\hat{K}$  are in involution at  $(x, J) = 0$ , i.e. on the cotangent bundle to the unit sphere  $T^*\mathbb{S}^2$ , and also on the zero level set of the second Hamiltonian.

The proof is a straightforward calculation.

**Proposition 5** *If  $A$  is an arbitrary symmetric matrix defining two functions on  $T^*\mathbb{R}^3$*

$$\tilde{H} = (x, Ax)\hat{H} - (x, x)(x, AJ)^2 = \begin{vmatrix} (x, x) & (x, Ax) & 0 \\ (J, J) & (J, AJ) & \sqrt{2}(x, AJ) \\ 0 & \sqrt{2}(x, AJ) & (x, Ax) \end{vmatrix} \quad (3.16)$$

and

$$\tilde{K} = (x, AJ)^2 \left( (x, Ax)(J, AJ) - (x, AJ)^2 \right) = [x, p, Ax]^2 \begin{vmatrix} (x, Ax) & (x, AJ) \\ (x, AJ) & (J, AJ) \end{vmatrix},$$

where  $[x, p, Ax]^2$  is a Gram determinant, then the Poisson brackets (3.14) between  $\tilde{H}$  and  $\tilde{K}$  is equal to

$$\{\tilde{H}, \tilde{K}\} = (x, J)(x, Ax)^2(x, AJ) \cdot \left( (J, AJ)\{(J, AJ), (x, Ax)\} + 4(x, AJ)\{(J, AJ), (x, AJ)\} \right).$$

Hence,  $\tilde{H}$  and  $\tilde{K}$  are in involution at  $(x, J) = 0$ , i.e. on the cotangent bundle to the unit sphere  $T^*\mathbb{S}^2$ , and also on the zero level set of the second Hamiltonian.

The proof is a straightforward calculation.

In elliptic coordinates on the sphere these Hamiltonians have the form (3.9-3.10) and (3.5-3.6) and, therefore, we have two bi-Hamiltonian system on  $T^*\mathbb{S}^2$ . Using natural Hamiltonians  $\hat{H}$  and  $\tilde{H}$  we can get globally defined integrable metrics and geodesic flows on the sphere  $\mathbb{S}^2$  [3, 4] whose integrals are polynomials in redundant momenta of degree four. We believe that our bi-Hamiltonian systems does not overlap with the implicit systems discussed in [21, 37, 40]. We can try to add some integrable potentials to the Hamiltonians  $\hat{H}$  and  $\tilde{H}$ , for instance potential (3.13), and try to obtain multidimensional generalizations of these bi-Hamiltonian systems. Discussion of these topics to look beyond the main content of this paper.

## 4 Conclusion

We study symmetries of the integrals of motion associated with the nonholonomic Veselova system using well-known divisor arithmetic on hyperelliptic curve of genus two [10]. In addition to the standard one-parametric auto Bäcklund transformations [14, 23] we also discuss symmetries related to a general divisor doubling [11] in order to get canonical transformation of valence two. Then we use these discrete symmetries in order to get new canonical variables on the phase space and new integrable Hamiltonian and conformally Hamiltonian systems in the framework of the Jacobi method.

With a pure mathematical point of view nonholonomic Veselova system is equivalent to the nonholonomic Chaplygin ball [29, 30]. It allows us to get new integrable deformations of the Chaplygin ball with integral of motion of fourth order in momenta and try to describe the corresponding physical model. We plan to study symmetries of integrals of motion and integrable deformations of the Chaplygin ball in a forthcoming publication.

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